

GEOMETRY OF REPRODUCING KERNELS IN MODEL SPACES NEAR THE BOUNDARY

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ABSTRACT. We study two geometric properties of reproducing kernels in model spaces K_θ where θ is an inner function in the disc: overcompleteness and existence of uniformly minimal systems of reproducing kernels which do not contain Riesz basic sequences. Both of these properties are related to the notion of the Ahern–Clark point. It is shown that “uniformly minimal non-Riesz” sequences of reproducing kernels exist near each Ahern–Clark point which is not an analyticity point for θ , while overcompleteness may occur only near the Ahern–Clark points of infinite order and is equivalent to a “zero localization property”. In this context the notion of quasi-analyticity appears naturally, and as a by-product of our results we give conditions in the spirit of Ahern–Clark for the restriction of a model space to a radius to be a class of quasi-analyticity.

1. INTRODUCTION AND MAIN RESULTS

Let $H^2 = H^2(\mathbb{D})$ denote the standard Hardy space in the unit disk \mathbb{D} , and let θ be an inner function in \mathbb{D} . The *model* (or *star-invariant*) *subspace* K_θ of H^2 is then defined as

$$K_\theta = H^2 \ominus \theta H^2.$$

According to the famous Beurling theorem, any closed subspace of H^2 invariant with respect to the backward shift in H^2 is of the form K_θ . For the numerous applications of model spaces in operator theory and in operator-related complex analysis see [29, 30].

Recall that the function

$$k_\lambda(z) = k_\lambda^\theta(z) = \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \overline{\lambda}z}$$

is the *reproducing kernel* for the space K_θ corresponding to the point $\lambda \in \mathbb{D}$, that is, $(f, k_\lambda^\theta) = f(\lambda)$ for any function $f \in K_\theta$. We usually omit the index θ when it is clear from the context which model space we consider. In what follows we denote by \tilde{k}_λ the normalized reproducing kernel, that is, $\tilde{k}_\lambda = k_\lambda / \|k_\lambda\|_2$.

Geometric properties of systems of reproducing kernels in model spaces is a deep and important subject which is studied extensively, see [21, 6, 27, 28, 7, 8, 9] for the study of completeness and [23, 20, 5, 11, 12, 10] for the results about bases of reproducing kernels. The main reason for that is that the geometric properties of reproducing kernels in a Hilbert

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space of analytic functions are related to the intrinsic analytic properties of the space. Let us mention several of such connections:

- a system of reproducing kernels $\{k_\lambda\}_{\lambda \in \Lambda}$ is complete in K_θ if and only if Λ is a uniqueness set for K_θ , i.e., if $f \in K_\theta$ vanishes on Λ , then $f = 0$;
- $\{\tilde{k}_\lambda\}_{\lambda \in \Lambda}$ is a Riesz basic sequence if and only if Λ is an *interpolating sequence*, i.e., for every data a_λ such that $\sum_{\lambda \in \Lambda} |a_\lambda|^2 \|k_\lambda\|^{-2} < \infty$, there exists a solution $f \in K_\theta$ of the interpolation problem $f(\lambda) = a_\lambda$, $\lambda \in \Lambda$;
- $\{\tilde{k}_\lambda\}_{\lambda \in \Lambda}$ is a Riesz basis if and only if Λ is a *complete interpolating sequence*, i.e., the above interpolation problem has a unique solution.

Another motivation relies on the fact that systems of eigenfunctions of certain second order differential equations are canonically unitarily equivalent to systems of reproducing kernels in model spaces (see [23] or [27]).

We are interested in the following two problems. The first of them, posed by Nikolai Nikolski, is related to the overcompleteness phenomenon in model spaces. Recall that a system of vectors $\{x_n\}$ in a separable Banach space X is said to be *overcomplete* if every subsequence $\{x_{n_k}\}$ is complete in X . Such sequences have for instance been discussed by Szegő who showed that the sequence $\frac{1}{t+\lambda_n}$ is overcomplete in $C([0, 1])$ whenever $\lambda_n \rightarrow \infty$ (see [17]). Klee [24] has shown that every separable Banach space possesses overcomplete sequences. For a system of reproducing kernels $\{k_{\lambda_n}\}$ in a reproducing kernel Hilbert space of functions analytic in some domain Ω , a trivial reason for being overcomplete is that λ_n tends to some point $\lambda_0 \in \Omega$. In most of the classical spaces (e.g., the Hardy or the Bergman space in the disc) there are no "nontrivial" overcomplete systems of reproducing kernels. However, in model spaces such systems may exist.

Problem 1. *Describe those inner functions θ for which there exists $\Lambda = \{\lambda_n\} \subset \mathbb{D}$ such that $\{k_\lambda\}_{\lambda \in \Lambda}$ is overcomplete in K_θ , but Λ has no accumulation points in the domain of analyticity of the elements of K_θ .*

This problem was explicitly addressed by Chalendar, Fricain and Partington in [15] (we refer also to [19]). In [15] it was shown that the overcompleteness for reproducing kernels in model spaces is related to the notion of the Ahern–Clark point for K_θ . Recall that for an inner function θ , the Ahern–Clark set is defined by

$$AC(\theta) = \left\{ \zeta \in \mathbb{T} : \sum_j \frac{1 - |z_j|^2}{|\zeta - z_j|^2} + \int_{\mathbb{T}} \frac{d\nu(z)}{|\zeta - z|^2} < \infty \right\},$$

where z_j are the zeros of θ (counting multiplicities) and ν is the singular measure generating its singular factor.

Recall that $AC(\theta)$ is exactly the set where θ has a unimodular non-tangential boundary value and a finite angular derivative. Such points are also referred to as Julia or Carathéodory points. Ahern and Clark showed in [2] that a function $f \in K_\theta$ has a finite non-tangential limit at a point $\zeta \in \mathbb{T}$ if and only if $\zeta \in AC(\theta)$. In this case the function k_ζ belongs to K_θ and is the reproducing kernel at the boundary point ζ . Note also that if we denote by $\sigma(\theta)$ the *boundary spectrum* of θ : $\sigma(\theta) = \{\zeta \in \mathbb{T} : \liminf_{z \rightarrow \zeta} |\theta(z)| = 0\}$, then θ

admits analytic continuation through $\mathbb{T} \setminus \sigma(\theta)$. Thus, clearly, $\mathbb{T} \setminus \sigma(\theta) \subset AC(\theta)$, however the most interesting situation for us is when $\zeta \in \sigma(\theta) \cap AC(\theta)$.

Concerning the overcompleteness problem, it is shown in [15] that if $\{k_\lambda\}_{\lambda \in \Lambda}$ is overcomplete, then (assuming $AC(\theta) \neq \mathbb{T}$)

$$\text{dist}(\Lambda, \mathbb{T} \setminus AC(\theta)) > 0.$$

For the proof see Corollary 2.3.

One of our main results says that the overcompleteness is equivalent to the *localization property* introduced recently by Abakumov, Belov and the first author in [1] in the context of de Branges spaces (i.e., essentially, in the case of model spaces K_θ such that $\sigma(\theta)$ consists of one point). Recall that the Stolz angle Γ_γ , $\gamma > 1$, at the point $\zeta \in \mathbb{T}$ is defined as $\Gamma_\gamma(\zeta) = \{z \in \mathbb{D} : |z - \zeta| \leq \gamma(1 - |z|)\}$.

Definition 1.1. *Let $\zeta \in \sigma(\theta)$. We say that the space K_θ has localization property at the point ζ , if any nonzero $f \in K_\theta$ has only finitely many zeros in any Stolz angle at ζ .*

In this definition, Stolz angle can be replaced by any region of the form $\{z \in \mathbb{D} : |z - \zeta|^N \leq \gamma(1 - |z|)\}$ for some $N, \gamma > 0$ (see Lemma 2.5). It should be observed that in our more general setting the different conditions of localization given in [1] are no longer equivalent. The very term of localization introduced in [1] means that the zeros of the functions in the space (the de Branges spaces they consider) are localized in certain regions. For our definition we pick the condition which, on the contrary, claims the existence of almost zero free regions (almost meaning up to a finite number). Still, our definition says that the zeros are localized outside a Stolz angle (or more general domains, see Lemma 2.5).

We can now state our first result.

Theorem 1.2. *Let $\zeta \in \mathbb{T}$. Then the following statements are equivalent:*

- (1) *There exists a sequence $\lambda_n \rightarrow \zeta$ such that $\{k_{\lambda_n}\}$ is overcomplete.*
- (2) *K_θ has the localization property at ζ .*

Moreover, if these conditions are satisfied, then $\zeta \in AC(\theta)$.

The most interesting case of this theorem is when $\zeta \in \sigma(\theta)$ since outside the spectrum of θ every function $f \in K_\theta$ has analytic continuation which immediately gives localization and overcompleteness.

Theorem 1.2 solves Problem 1. However, the points with localization do not admit an explicit description. In some special situations such descriptions are given in [1] where the relations of the localization property in de Branges spaces with the structure of its subspaces and with the spectral theory of canonical systems is revealed.

Now we state two corollaries of Theorem 1.2. The first result provides a necessary condition for overcompleteness and extends significantly the results of [15]. The Ahern–Clark set of higher order $AC_n(\theta)$ is defined analogously to $AC(\theta)$ (see Section 2) and is related to the existence of non-tangential boundary values of derivatives in K_θ [2].

Theorem 1.3. *If $\lambda_n \rightarrow \zeta \in \mathbb{T}$ and $\{k_{\lambda_n}\}$ is overcomplete then $\zeta \in \bigcap_{n=0}^{\infty} AC_n(\theta)$.*

The converse of this result is not true as will be shown by the example given in Section 3 of a point $\zeta \in \bigcap_{n=0}^{\infty} AC_n(\theta)$ which is not a point of localization (and, thus, there is no overcomplete sequence $\{k_{\lambda_n}\}$ with $\lambda_n \rightarrow \zeta$).

Next we introduce the notion of strong localization which will turn out to be a sufficient condition for overcompleteness at a point of the boundary spectrum of θ (it seems that no such examples were given in [15]). Recall that with each $\alpha \in \mathbb{T}$ we can associate a singular measure σ_α on \mathbb{T} , the so-called Clark measure (see Section 2 for details). The condition $\zeta \in AC(\theta)$ is equivalent to the fact that $\sigma_\alpha(\{\zeta\}) \neq 0$ for some (unique) $\alpha \in \mathbb{T}$.

Definition 1.4. *Let $\zeta \in \bigcap_{n \geq 0} AC_n(\theta)$. We say that K_θ has strong localization at ζ , if the system $\{(z - \zeta)^{-k}\}_{k \geq 1}$ is complete in $L^2(\sigma_\alpha)$ for some (any) $\alpha \in \mathbb{T}$ such that $\sigma_\alpha(\{\zeta\}) = 0$.*

Again we should emphasize that we are in a more general situation than in [1]. In that paper strong localization gave a more precise information on the localization of the zeros (namely, each zero except a finite number was located near a point mass of the Clark measure σ_α). As it turns out in de Branges spaces considered in [1] this is equivalent to the density definition given above. A priori, in our setting it is not immediately clear why strong localization should imply localization. That this is actually the case will be discussed below in Corollary 1.7.

Interestingly, the above definition, which may look rather abstract at first sight, can be connected with another well known property, namely that of quasi-analyticity.

Theorem 1.5. *The point $\zeta = 1$ is a point of strong localization for K_θ if and only if $K_\theta|_{[0,1]}$ is a class of quasi-analyticity.*

Since quasi-analyticity is a stronger requirement than just being C^∞ -smooth, this theorem leads naturally to the question whether it is possible to characterize quasi-analyticity of $K_\theta|_{[0,1]}$ in the spirit of Ahern–Clark, i.e. in terms of the behavior of the zeros and the singular measure of θ near ζ .

Using some classical results on polynomials approximation, it is possible to give the the following sufficient condition in terms of the Clark measure which is in the spirit of another way of characterizing Ahern–Clark points of arbitrary order (cf. (5) below).

Theorem 1.6. *Let K_θ be a model space in the disc and let σ be some Clark measure for K_θ . Assume that, for some $\varepsilon > 0$,*

$$(1) \quad \int_{\mathbb{T}} \exp\left(\frac{\varepsilon}{|\eta - \zeta|}\right) d\sigma(\eta) < \infty.$$

Then ζ is a strong localization point for K_θ , i.e., $K_\theta|_{[0,\zeta]}$ is a class of quasi-analyticity.

Unfortunately, and contrarily to the Ahern–Clark situation, it cannot be expected that a condition of type (1) with the exponential replaced by some appropriate function is necessary and sufficient for strong localization or quasi-analyticity. We will discuss this matter more thoroughly through the results of Borichev and Sodin [13] after the proof of Theorem 1.6 in Section 4.

With Theorem 1.5 in mind, we are able to deduce the following consequence.

Corollary 1.7. *Strong localization at $\zeta \in \mathbb{T}$ implies localization at ζ .*

This corollary will automatically provide a sufficient condition for localization at $\zeta \in \mathbb{T}$ and thus existence of overcomplete systems accumulating to ζ . Examples are given in [1] showing that there are points of localization which are not of strong localization.

Let us now turn to the second problem considered in the present paper.

Problem 2. *Describe those inner functions for which there exists a uniformly minimal sequence $\{\tilde{k}_\lambda\}_{\lambda \in \Lambda}$ which does not contain any Riesz sequence.*

Such systems of reproducing kernels will be called UMNR systems (i.e., uniformly minimal non-Riesz). Note that again most of the classical spaces of analytic functions do not possess such systems. Also there exist model spaces for which the class UMNR is empty. This is for instance the case for K_θ with $\theta(z) = \exp\left(a\frac{1+z}{1-z}\right)$, $a > 0$, or for the corresponding model space in the upper half-plane associated with the inner function $\Theta(z) = \exp(iaz)$. Indeed, in this case it is known that any incomplete system of normalized reproducing kernels contains Riesz sequences.

It turns out that UMNR systems of reproducing kernels in K_θ exist if and only if θ has "nontrivial" Ahern–Clark points: $\sigma(\theta) \cap AC(\theta) \neq \emptyset$.

Theorem 1.8. *Let $\zeta \in AC(\theta) \cap \sigma(\theta)$. Then there exists a sequence $\lambda_n \rightarrow \zeta$ such that $\{\tilde{k}_{\lambda_n}\}$ is UMNR.*

Note that an overcomplete system is never uniformly minimal. Still, an overcomplete system can be minimal. A related result concerns the possibility of extracting a uniformly minimal system from a minimal system. The following observation will follow from our discussions in a rather simple way.

Theorem 1.9. *Let $\zeta \in AC(\theta)$ which is not a point of localization. Let $\{z_n\}$ be a sequence in \mathbb{D} . Then*

- (1) *If $z_n \rightarrow \zeta \in \mathbb{T}$ non-tangentially then $\{\tilde{k}_{z_n}\}$ is a minimal sequence that does not contain any uniformly minimal sequence.*
- (2) *If $z_n \rightarrow \zeta$, but $\|k_{z_n} - k_\zeta\| \not\rightarrow 0$, then $\{\tilde{k}_{z_n}\}$ contains a uniformly minimal sequence.*

Note that from Theorems 1.2 and 1.3 it is easily seen that if $\zeta \in AC(\theta) \setminus AC_1(\theta)$ then we are in the setting of the above theorem.

A final word concerning notation. In this paper the notation $U(z) \lesssim V(z)$ means that there is a constant $C > 0$ such that $U(z) \leq CV(z)$ holds for all suitable values of the variable z . We write $U(z) \asymp V(z)$ if $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2. PRELIMINARIES

2.1. Necessity of the Ahern–Clark condition. We start with some simple observations on the geometry of vectors in Hilbert spaces. We write $x_n \xrightarrow{w} x_0$ if the sequence x_n converges weakly to x_0 in a Hilbert space H .

Lemma 2.1. *A normalized sequence $\{x_n\}$ in a Hilbert space contains a Riesz sequence if and only if it contains a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{w} 0$.*

Proof. Clearly, if $\{x_{n_k}\}$ is a Riesz sequence, then $x_{n_k} \xrightarrow{w} 0$. Conversely, if $\{x_n\}$ contains a subsequence weakly converging to zero, then, proceeding inductively, we can choose a subsequence $\{x_{n_k}\}$ such that

$$(2) \quad \sum_{\ell} \sum_{k \neq \ell} |(x_{n_k}, x_{n_\ell})|^2 < 1.$$

Then, denoting by $G = (x_{n_k}, x_{n_\ell})$ the Gram matrix associated with $\{x_{n_k}\}$, and writing $G = Id + G_0$ we see that (2) implies that G is bounded and invertible, whence x_{n_k} is a Riesz sequence (see [30, Volume 2, p.171]). \square

In the next corollary we will use the following fact: for any inner function θ the set $K_\theta \cap C(\overline{\mathbb{D}})$ is dense in K_θ . While this is trivial when θ is a Blaschke product, it is in general a nontrivial fact due to Aleksandrov [3].

Corollary 2.2. *The sequence of normalized reproducing kernels $\{\tilde{k}_\lambda\}_{\lambda \in \Lambda}$ in K_θ contains a Riesz subsequence if and only if $\sup_{\lambda \in \Lambda} \|k_\lambda\| = \infty$.*

Proof. Without loss of generality, let $\|k_{\lambda_n}\| \rightarrow \infty$, $n \rightarrow \infty$. For any $f \in K_\theta \cap C(\overline{\mathbb{D}})$,

$$(f, \tilde{k}_{\lambda_n}) = \frac{f(\lambda_n)}{\|k_{\lambda_n}\|} \rightarrow 0.$$

Hence, $\tilde{k}_{\lambda_n} \xrightarrow{w} 0$, and it suffices to apply Lemma 2.1. The converse statement is immediate. \square

Corollary 2.3. *If $\lambda_n \rightarrow \zeta \in \mathbb{T}$ and $\{\tilde{k}_{\lambda_n}\}$ is overcomplete or UMNR, then $\sup_n \|k_{\lambda_n}\| < \infty$, whence $\zeta \in AC(\theta)$.*

Proof. In both situations $\{\tilde{k}_{\lambda_n}\}$ does not contain any Riesz sequence, so that by the preceding corollary we have $\sup_n \|k_{\lambda_n}\| < \infty$. Recall that $\|k_{\lambda_n}\|^2 = \frac{1 - |\theta(\lambda_n)|^2}{1 - |\lambda_n|^2}$, whence

$$\limsup_{n \rightarrow \infty} \frac{1 - |\theta(\lambda_n)|^2}{1 - |\lambda_n|^2} < \infty.$$

Now the classical Julia–Carathéodory theorem implies that $\zeta \in AC(\theta)$. \square

2.2. Higher order Ahern–Clark condition and Clark measures. Let z_j be the zeros of an inner function θ (counting multiplicities) and let ν be the singular measure generating its singular factor. We say that $\zeta \in \mathbb{T}$ is in $AC_n(\theta)$, the Ahern–Clark set of order n , if

$$(3) \quad \sum_j \frac{1 - |z_j|^2}{|\zeta - z_j|^{2n+2}} + \int_{\mathbb{T}} \frac{d\nu(z)}{|\zeta - z|^{2n+2}} < \infty.$$

By the results of Ahern–Clark, $\zeta \in AC_n(\theta)$ if and only if there exist non-tangential limits of $f^{(k)}(z)$, $0 \leq k \leq n$, as $z \rightarrow \zeta$, for every $f \in K_\theta$. Note that in this notation $AC(\theta) = AC_0(\theta)$.

Recall that the measure σ_α , $\alpha \in \mathbb{T}$, from the representation

$$\frac{\alpha + \theta(z)}{\alpha - \theta(z)} = \int_{\mathbb{T}} \frac{1 + \bar{\xi}z}{1 - \bar{\xi}z} d\sigma_\alpha(\xi)$$

is called the Clark measure for K_θ (see [16]). We sometimes write σ_α^θ to emphasize the dependence on θ . Any function $f \in K_\theta$ has non-tangential boundary values σ_α -everywhere [31], $\|f\| = \|f\|_{L^2(\sigma_\alpha)}$, and the map

$$(4) \quad V : h \in L^2(\sigma_\alpha) \mapsto f(z) = (\alpha - \theta(z)) \int_{\mathbb{T}} \frac{h(\xi)}{1 - \bar{\xi}z} d\sigma_\alpha(\xi)$$

is a unitary map from $L^2(\sigma_\alpha)$ onto K_θ . If $\zeta \in AC(\theta)$, then there exists α_0 such that $\sigma_{\alpha_0}(\{\zeta\}) > 0$. It is well known (see, e.g., [32, VII-2]) that

$$(5) \quad \zeta \in AC_n(\theta) \iff \int_{\mathbb{T}} \frac{d\sigma_\alpha(\eta)}{|1 - \bar{\eta}\zeta|^{2n+2}} < \infty, \quad \alpha \neq \alpha_0.$$

2.3. Transfer to the upper half-plane. In what follows it will be often convenient to pass to an equivalent problem in the half-plane setting where the estimates and computations become much simpler. For $\zeta \in \mathbb{T}$, consider the conformal mapping

$$(6) \quad w(z) = i \frac{\zeta + z}{\zeta - z},$$

which maps \mathbb{D} onto the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$, the unit circle $\mathbb{T} \setminus \{\zeta\}$ onto the real axis \mathbb{R} , and the point ζ to ∞ . For an inner function θ in \mathbb{D} , put $\Theta(w) = \theta(\zeta \frac{w-i}{w+i})$. Then Θ is an inner function in \mathbb{C}^+ .

It is well known that

$$T : f \rightarrow \frac{1}{w+i} f\left(\zeta \frac{w-i}{w+i}\right)$$

is a unitary mapping from $H^2(\mathbb{D})$ to the Hardy space $H^2(\mathbb{C}^+)$ in the upper half-plane and $TK_\theta = K_\Theta = H^2(\mathbb{C}^+) \ominus \Theta H^2(\mathbb{C}^+)$ (see [30, Chapter A6]).

The following property of the spaces K_Θ will often be used in what follows: given $f \in H^2(\mathbb{C}^+)$,

$$(7) \quad f \in K_\Theta \iff \overline{f(t)}\Theta(t) \in H^2(\mathbb{C}^+),$$

which means that the function $\overline{f(t)}\Theta(t)$ on \mathbb{R} coincides with the non-tangential boundary values of some function in $H^2(\mathbb{C}^+)$.

Let ν_0 be a measure on \mathbb{T} . Then the change of variable

$$d\nu_0(\tau) = \frac{d\nu(t)}{t^2 + 1}, \quad t \in \mathbb{R}, \quad \tau = \zeta \frac{t-i}{t+i} \in \mathbb{T},$$

gives us a measure ν on \mathbb{R} . The Ahern–Clark conditions of order n for the point infinity may then be rewritten in terms of the zeros $w_j = x_j + iy_j$ of Θ and of the corresponding

singular measure ν as follows:

$$(8) \quad \infty \in AC_n(\Theta) \iff \sum_j y_j(1 + |w_j|^2)^n + \int_{\mathbb{R}} (1 + x^2)^n d\nu(x) < \infty,$$

while in terms of the Clark measures σ_α for Θ (note that we use the same notation) the Ahern–Clark condition of order n becomes

$$(9) \quad \infty \in AC_n(\Theta) \iff \int_{\mathbb{R}} (1 + x^2)^n d\sigma_\alpha < \infty$$

for all $\alpha \in \mathbb{T}$, $\alpha \neq \alpha_0 = \lim_{y \rightarrow +\infty} \Theta(iy)$. In particular, the usual Ahern–Clark condition (of order 0) means that $\sigma_\alpha(\mathbb{R}) < \infty$.

Now we state the localization and strong localization properties at the point ∞ .

Definition 2.4. *The space K_Θ in \mathbb{C}^+ has localization property at the point ∞ , if any nonzero $f \in K_\Theta$ has only finitely many zeros in any Stolz angle $\Gamma_\gamma = \{z \in \mathbb{C}^+ : |z| > 1 \text{ and } \operatorname{Im} z \geq \gamma |\operatorname{Re} z|\}$, $\gamma > 0$.*

The space K_Θ has the strong localization at ∞ if for any Clark measure σ_α except $\alpha = \lim_{y \rightarrow \infty} \Theta(iy)$ the polynomials belong to the space $L^2(\sigma_\alpha)$ and are dense there.

Note that in this definition we have added the condition $|z| > 1$ in order to distinguish Γ_ζ from the Stolz angle at 0.

Both of the above definitions are equivalent to the localization (strong localization) property at ζ for the function θ related to the function Θ by (6).

In the half-plane setting it is easy to see, using an idea from [1], that in the definition of the localization at ∞ the Stolz angle may be replaced by any domain of the form $\Gamma_{\gamma,\beta} = \{\operatorname{Im} z > \gamma |\operatorname{Re} z|^\beta, |z| > 1\}$ where $\gamma > 0$, $\beta \in \mathbb{R}$.

Lemma 2.5. *If K_Θ has the localization property at ∞ , then any nonzero $f \in K_\Theta$ has only a finite number of zeros in any domain $\Gamma_{\gamma,\beta}$.*

Proof. Assume the converse and let $f \in K_\Theta$ have infinitely many zeros in some domain $\Gamma_{\gamma,\beta}$. Choose a subsequence $\{\lambda_n\}$ of such zeros such that $|\lambda_{n+1}| > 2|\lambda_n|$. Then the infinite product $G(z) = \prod_n (1 - z/\lambda_n)$ converges and $\lim_{|x| \rightarrow \infty} |x|^{-N} |G(x)| = \infty$ for any $N > 0$ (the limit is taken over $x \in \mathbb{R}$). Here we use the fact that

$$|1 - x/\lambda_n| \geq |x - \lambda_n|/|\lambda_n| \geq |\operatorname{Im} \lambda_n|/|\lambda_n| \geq \gamma |\lambda_n|^{-|\beta|-1},$$

and the lacunarity of $\{\lambda_n\}$.

Now we may choose a sequence iy_n which is so sparse that the infinite product $\tilde{G}(z) = \prod_n (1 - z/iy_n)$ converges and $|\tilde{G}(x)| \leq C|G(x)|$ on \mathbb{R} for some $C > 0$ (e.g., take $y_n = \lambda_{10n}$). Then $g(z) = \tilde{G}(z)f(z)/G(z)$ is in $H^2(\mathbb{C}^+)$ and also

$$\overline{g(t)}\Theta(t) = \overline{f(t)}\Theta(t)\tilde{G}^*(t)/G^*(t) \in H^2(\mathbb{C}^+),$$

where $G^*(z) = \overline{G(\bar{z})}$. So, by (7), $g \in K_\Theta$ and $g(iy_n) = 0$, a contradiction to the localization at ∞ . \square

3. OVERCOMPLETENESS AND LOCALIZATION

In this section we give the proofs of Theorems 1.2 and 1.3. For this we need one more equivalent form of localization.

Proposition 3.1. *If $\zeta \in \mathbb{T}$ is not a point of localization for K_θ , then for any sequence $\lambda_n \rightarrow \zeta$ there exist a subsequence λ_{n_k} and $f \in K_\theta$, $f \neq 0$, such that $f(\lambda_{n_k}) = 0$.*

The converse is trivially true. Observe from Lemma 2.5 that localization is only determined by the behavior of zeros inside Stolz domains or their generalized form $\Gamma_{\gamma,\beta}$.

Proof. Pass to \mathbb{C}^+ by the conformal mapping (6) which maps ζ to ∞ . The condition that there is a function with infinitely many points on the radius means now that there exists $f \in K_\theta$ and $y_n \rightarrow +\infty$ such that $f(iy_n) = 0$. Let $\{\lambda_n\}$ be any sequence tending to infinity. Let us choose a lacunary product $E = \prod(1 - z/iy_n)$. We can always choose an even more lacunary product $G = \prod(1 - z/\lambda_{n_k})$ with $\lambda_{n_k} \in \{\lambda_n\}$ such that $|G(x)| \leq |E(x)|$ on \mathbb{R} . Then, making use of (7), it is easy to see that $\tilde{f}(z) = f(z)G(z)/E(z)$ will belong to K_θ and vanish on $\{\lambda_{n_k}\}$. \square

Proof of Theorem 1.2. (2) \implies (1) is trivial, any λ_n which tends to ζ along the radius gives an overcomplete system.

(1) \implies (2) follows from Proposition 3.1. \square

Proof of Theorem 1.3. Assume that there exists an overcomplete system $\{k_{\lambda_n}\}$ with $\lambda_n \rightarrow \zeta$, but $\zeta \notin AC_n(\theta)$ and $\zeta \in AC_{n-1}(\theta)$ for some $n \geq 1$. Note that we already know from Corollary 2.3 that necessarily $\zeta \in AC(\theta) = AC_0(\theta)$. By (5), there exists $\alpha \in \mathbb{T}$ such that

$$\int_{\mathbb{T}} \frac{d\sigma_\alpha(\eta)}{|1 - \bar{\eta}\zeta|^{2n}} = \infty \quad \text{and} \quad \int_{\mathbb{T}} \frac{d\sigma_\alpha(\eta)}{|1 - \bar{\eta}\zeta|^{2k}} < \infty, \quad k < n.$$

Passing to \mathbb{C}^+ by the conformal mapping (6), we get a space K_Θ in \mathbb{C}^+ with a Clark measure $\mu = \sigma_\alpha^\Theta$ such that

$$\int_{\mathbb{R}} |t|^{2n} d\mu(t) = \infty \quad \text{and} \quad \int_{\mathbb{R}} |t|^{2k} d\mu(t) < \infty, \quad k < n$$

(see the discussion in Subsection 2.3). Consider the measure $d\tilde{\mu}(t) = |t|^{2n} d\mu(t)$. We thus have $\tilde{\mu}(\mathbb{R}) = \infty$, but $\int_{\mathbb{R}} \frac{d\tilde{\mu}(t)}{t^2 + 1} < \infty$. Define an inner function $\tilde{\Theta}$ in \mathbb{C}^+ by the formula

$$(10) \quad i \frac{1 + \tilde{\Theta}(z)}{1 - \tilde{\Theta}(z)} = \int \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\tilde{\mu}(t).$$

Then, clearly, $\tilde{\mu} = \sigma_1^{\tilde{\Theta}}$, the Clark measure for $K_{\tilde{\Theta}}$.

Note that the model space $K_{\tilde{\Theta}}$ has no localization at ∞ . Indeed, if $1 \neq \lim_{y \rightarrow +\infty} \tilde{\Theta}(iy)$, then $\infty \notin AC(\tilde{\Theta})$ by (9), since $\tilde{\mu}(\mathbb{R}) = \infty$. If $1 = \lim_{y \rightarrow +\infty} \tilde{\Theta}(iy)$ and $\infty \in AC(\tilde{\Theta})$, then

by definition of the angular derivative at ∞ we must have $0 < \lim_{y \rightarrow \infty} y(1 - \tilde{\Theta}(iy)) < \infty$ so that

$$\lim_{y \rightarrow \infty} \frac{1}{y} \left| \frac{1 + \tilde{\Theta}(iy)}{1 - \tilde{\Theta}(iy)} \right| > 0.$$

However, it follows from (10) that the above limit is zero. We conclude that ∞ is not an Ahern–Clark point for $\tilde{\Theta}$ and, thus, not a localization point for $K_{\tilde{\Theta}}$.

We will now use the unitary operator $V_+ : L^2(\tilde{\mu}) \rightarrow K_{\tilde{\Theta}}$ already mentioned earlier, which in the half-plane setting is defined by $V_+ f(z) = (1 - \tilde{\Theta}(z)) \int_{\mathbb{R}} \frac{u(t)}{t-z} d\tilde{\mu}(t)$. By Theorem 1.2, there exists $u \in L^2(\tilde{\mu})$ such that the function $h \in K_{\tilde{\Theta}}$ defined by

$$h(z) = (1 - \tilde{\Theta}(z)) \int_{\mathbb{R}} \frac{u(t)}{t-z} d\tilde{\mu}(t) = (1 - \tilde{\Theta}(z)) \int_{\mathbb{R}} \frac{u(t)t^{2n}}{t-z} d\mu(t)$$

has infinitely many (simple) zeros of the form $\{iy_m\}$, $y_m \rightarrow \infty$. Moreover, note that the functions $\varphi_m(z) := \frac{h(z)}{z - iy_m}$ belong to $K_{\tilde{\Theta}}$, vanish at iy_ℓ , $\ell \neq m$, and are linearly independent. Write $\varphi_m(z) = V_+ u_m$. Clearly for an appropriate finite linear combination v of u_m , we achieve

$$\int_{\mathbb{R}} v(t)t^k d\mu(t) = 0, \quad k \leq 2n-1.$$

By construction, the function

$$g(z) = \int_{\mathbb{R}} \frac{v(t)t^{2n}}{t-z} d\mu(t)$$

vanishes at iy_m for m sufficiently big (i.e., $m \geq m_0$).

Now let

$$f(z) = \int_{\mathbb{R}} \frac{v(t)}{t-z} d\mu(t).$$

Clearly, $v \in L^2(\mu)$ and so $(\alpha - \Theta)f \in K_{\Theta}$ (recall that $\mu = \sigma_{\alpha}^{\Theta}$). Let us show that $f(iy_m) = 0$ for m sufficiently big. Indeed, using $1 = t^{2n}z^{-2n} - (t-z) \sum_{k=0}^{2n-1} t^k z^{-k-1}$, we can write

$$f(z) = - \sum_{k=0}^{2n-1} \frac{1}{z^{k+1}} \underbrace{\int_{\mathbb{R}} v(t)t^k d\mu(t)}_0 + \frac{1}{z^{2n}} \underbrace{\int_{\mathbb{R}} \frac{v(t)t^{2n}}{t-z} d\mu(t)}_{g(z)}.$$

Hence, $f(iy_m) = 0$, $m > m_0$, which contradicts the fact that ∞ is a localization point for K_{Θ} . \square

Example 3.2. The converse is not true: there exist points $\zeta \in \bigcap_{n=0}^{\infty} AC_n(\theta)$ which are not points of localization for K_{θ} . In view of the conformal mapping, it is sufficient to construct a Blaschke product B in \mathbb{C}^+ such that $\infty \in \bigcap_{n=0}^{\infty} AC_n(B)$ but ∞ is not a localization point for K_B .

Let B be the Blaschke product with zeros

$$z_n = |n|^{\alpha} \text{sign } n + i \exp(-|n|^{1/\beta}), \quad n \in \mathbb{Z},$$

where $1 < \alpha < \beta$. Put $E(z) = \prod_n (1 - z/\bar{z}_n)$. It is then clear that $B = \gamma E^*/E$ for some unimodular constant γ (recall that we define $g^*(z) = \overline{g(\bar{z})}$). By (8) we have $\infty \in \bigcap_{n=0}^\infty AC_n(B)$.

By standard estimates of canonical products (see, e.g., [26, Ch. 2]) we have

$$\log \left| \frac{E(z)}{\text{dist}(z, \{\bar{z}_n\})} \right| \asymp |z|^{1/\alpha}, \quad |z| > 1,$$

and in particular, $\log |E(x)| \asymp |x|^{1/\alpha}$, $x \in \mathbb{R}$, $|x| \geq 1$. Let $F(z)$ be an entire function of order less than $1/\alpha$ with imaginary zeros, say, $F(z) = \prod_n (1 - z/(2^n i))$. Clearly, $F/E \in L^2(\mathbb{R})$ and, hence, $f = F/E \in H^2(\mathbb{C}^+)$, since any entire function of order less than 1 is of Smirnov class in the upper half-plane. Also, as in the proof of Lemma 2.5,

$$\overline{f(t)}B(t) = \frac{\overline{F(t)}}{\overline{E(t)}} \cdot \frac{\overline{E(t)}}{E(t)} = \frac{F^*(t)}{E(t)}, \quad t \in \mathbb{R}.$$

By similar reasons as above, $F^*/E \in H^2(\mathbb{C}^+)$ whence, by (7), $f \in K_B$. Since f has infinitely many imaginary zeros, we conclude that ∞ is not a localization point for K_B .

4. STRONG LOCALIZATION AND QUASI-ANALYTICITY

Proof of Theorem 1.5. Observe first that strong localization requires by definition that $1 \in \bigcap_{n \geq 0} AC_n(\theta)$, and if $K_\theta|_{[0,1]}$ is a class of quasi-analyticity then all the derivatives of $f \in K_\theta$ are supposed to exist radially so that $1 \in \bigcap_{n \geq 0} AC_n(\theta)$, and we can implicitly admit this condition.

Recall that if $1 \in \bigcap_{n \geq 0} AC_n(\theta)$ then $(z-1)^{-n} \in L^2(\sigma)$ for every $n \in \mathbb{N}$ (see for instance [32, VII-2]). Necessarily in this case $\sigma(\{1\}) = 0$. Again we will use the fact that $K_\theta = VL^2(\sigma)$, where σ is the Clark measure that we suppose associated with $\alpha = 1$, and the isometry V is defined by (4).

Suppose 1 is a point of strong localization. Pick an arbitrary function $f = Vh \in K_\theta$, and suppose that $f^{(n)}(1) = 0$ for every $n \in \mathbb{N}^*$. In order to show that $K_\theta|_{[0,1]}$ is a class of quasi-analyticity we have to check that f vanishes identically.

Since f has a zero of arbitrary order at 1, the function g defined by $g(z) = f(z)/(1-\theta(z))$ has also a zero of arbitrary order at 1 (note that θ has the same regularity at 1 as any function in K_θ , and $\lim_{r \rightarrow 1} \theta(r) \neq 1$). So $\lim_{r \rightarrow 1} g^{(n)}(r) = g^{(n)}(1) = 0$ for every $n \in \mathbb{N}$. Clearly

$$g^{(n)}(z) = \frac{d^n}{dz^n} \int_{\mathbb{T}} \frac{h(\zeta)}{1 - \bar{\zeta}z} d\sigma(\zeta) = n! \int_{\mathbb{T}} \frac{\bar{\zeta}^n h(\zeta)}{(1 - \bar{\zeta}z)^{n+1}} d\sigma(\zeta).$$

Observe that

$$\left| \frac{\bar{\zeta}^n h(\zeta)}{(1 - \bar{\zeta}z)^{n+1}} \right| \lesssim \frac{|h(\zeta)|}{|1 - \bar{\zeta}|^{n+1}}.$$

The function on the right hand side is integrable since $h \in L^2(\sigma)$ and $(1-z)^{-k} \in L^2(\sigma)$ for every k . Since we also have pointwise convergence, by Lebesgues' dominated convergence theorem we conclude

$$0 = \lim_{r \rightarrow 1} g^{(n)}(r) = n! \int_{\mathbb{T}} \frac{\bar{\zeta}^n h(\zeta)}{(1-\bar{\zeta})^{n+1}} d\sigma(\zeta).$$

It remains to use an inductive argument. For $n = 0$, we conclude that $h \perp (1-\zeta)^{-1}$ (with respect to the scalar product in $L^2(\sigma)$). Suppose $h \perp (1-\zeta)^{-k}$ for $1 \leq k \leq n$. Note that

$$\frac{\bar{\zeta}^n}{(1-\bar{\zeta})^{n+1}} = \frac{1}{(1-\bar{\zeta})^{n+1}} - \frac{(1-\bar{\zeta}^n)}{(1-\bar{\zeta})^{n+1}} = \frac{1}{(1-\bar{\zeta})^{n+1}} - \frac{(1+\bar{\zeta}+\dots+\bar{\zeta}^{n-1})}{(1-\bar{\zeta})^n}.$$

Since $\frac{(1+\bar{\zeta}+\dots+\bar{\zeta}^{n-1})}{(1-\bar{\zeta})^n}$ is in the space generated by $(1-\zeta)^{-k}$, $1 \leq k \leq n$, integrating against h in the last term with respect to $d\sigma$ yields 0. Hence

$$\int_{\mathbb{T}} \frac{h(\zeta)}{(1-\bar{\zeta})^{n+1}} d\sigma(\zeta) = 0,$$

which achieves the induction. We have thus proved that if the function f vanishes to arbitrary order at 1, then $h \perp (1-\zeta)^{-n}$ for every $n \in \mathbb{N}^*$. By strong localization, these functions generate the whole space $L^2(\sigma)$, so that $h = 0$, and hence $f = 0$.

For the converse, the argument is almost the same. Suppose $K_\theta|_{[0,1]}$ is a class of quasi-analyticity. Pick any $h \in L^2(\sigma)$ and suppose $h \perp (1-\zeta)^{-n}$, $n \in \mathbb{N}^*$. By construction $f = Vh \in K_\theta$, and, associating with this f the function g as above, we notice that

$$\lim_{r \rightarrow 1} g^{(n)}(r) = n! \int_{\mathbb{T}} \frac{\bar{\zeta}^n h(\zeta)}{(1-\bar{\zeta})^{n+1}} d\sigma(\zeta) = 0$$

(again observe that $\zeta^n/(1-\zeta)^{n+1}$ is in the space generated by $(1-\zeta)^{-k}$, $1 \leq k \leq n+1$). Thus $f = (1-\theta)g$ has zero of arbitrary order at 1, in other words $f^{(n)}(1) = 0$ for every $n \in \mathbb{N}$. By quasi-analyticity f has to vanish on $[0,1]$ and thus on \mathbb{D} , which implies that $h = 0$. We conclude that $(1-\zeta)^n$, $n \in \mathbb{N}^*$, generates a dense subspace. \square

Proof of Corollary 1.7. We still suppose $\zeta = 1$ for simplicity. By Theorem 1.5, strong localization is equivalent to quasi-analyticity.

Recall also that we can again assume $1 \in \bigcap_{n \geq 0} AC_n(\theta)$.

Now suppose there is a function $f \in K_\theta$ with infinitely many zeros z_k in a Stolz angle at 1. Then in particular $\lim_{z \searrow 1} f(z) = \lim_{k \rightarrow +\infty} f(z_k) = 0$. Then also $\lim_{z \searrow 1} \frac{f(z)-f(1)}{z-1} = \lim_{k \rightarrow +\infty} \frac{f(z_k)}{z_k} = 0$. By induction we obtain that $f^{(n)}(1) = 0$ for every $n \in \mathbb{N}$. Since $K_\theta|_{[0,1]}$ is quasi-analytic, we conclude that f vanishes identically. \square

Proof of Theorem 1.6. Passing to an equivalent problem in the space K_Θ in the upper

half-plane (related to K_θ by (6)) and the point ∞ , we obtain a Clark measure $\mu = \sigma_\alpha^\theta$ for K_Θ such that

$$(11) \quad \int_{\mathbb{R}} e^{\varepsilon|t|} d\mu(t) < \infty.$$

As we have seen before, strong localization in the the upper half is related with weighted polynomial approximation which is one of the most classical subjects of analysis (for a detailed survey see [18, 25]). It is well known that under condition (11) the polynomials are dense in $L^2(\mu)$ (see, e.g., [18, Theorem II.5.2], or [30, Exercise A4.8.3(c)]), and so ∞ is a strong localization point for K_Θ . The value α is not exceptional for K_Θ since σ_α^θ has no point mass at ζ . \square

As already mentioned in the introduction, and contrarily to the Clark measure formulation (5) or (9) of the Ahern–Clark condition for existence of non-tangential higher order derivatives at boundary points, a condition of type $\int_{\mathbb{R}} \Phi(t) d\mu(t) < \infty$ (case of the line) cannot give a necessary and sufficient condition for completeness of polynomials, and hence quasi-analyticity. We will discuss this through the results of [13] as presented in [30, Exercise A4.8.3(ℓ)].

In order to do so, consider the sequence $\Lambda_\rho = \{n^{1/\rho} : n = 1, 2, \dots\}$, $\rho > 0$, and the weight $w_{m,s}(\lambda) = \lambda^s e^{-c\lambda^m}$, $m > 1$, $c > 1$, $s \in \mathbb{R}$. Set $\mu = \sum_{\lambda \in \Lambda_\rho} w_{m,s}^p(\lambda) \delta_\lambda$. This singular measure is finite and it is possible (after a possible normalization) to associate with it a model space K_Θ (we will consider the case $p = 2$ here).

According to [13], if $m \geq 1/2$, then the polynomials are always dense in

$$L^p(\mu) = \ell^p(\Lambda_\rho, w_{m,s}^p) = \{x = (x(\lambda))_{\lambda \in \Lambda_\rho} : \sum_{\lambda \in \Lambda_\rho} |x(\lambda) w_{m,s}(\lambda)|^p < \infty\}$$

(again, we are only interested in the case $p = 2$ here). However

$$\int_{\mathbb{R}} \Phi(t) d\mu(t) = \sum_{n \geq 1} \Phi(n^{1/\rho}) \frac{n^{ps/\rho}}{e^{c p n^{m/\rho}}}$$

converges if $\Phi(x) = O(e^{c p x^{m'}})$ for $m' < m$ and diverges if $\liminf_{x \rightarrow \infty} \Phi(x) e^{-c p x^{m'}} > 0$ for $m' \geq m$ (and $s > 0$). So, integrability against a function Φ cannot be necessary and sufficient.

Considering the case $0 < \rho = m < 1/2$, there exists a constant $c_0 = \pi \operatorname{ctg}(\pi \rho)$ such that if $c > c_0$, the polynomials are dense, and if $c < c_0$ they are not (there are also some discussions on the case $c = c_0$; see [13] or [30, Exercise A4.8.3(ℓ)] for all these results). In this situation the integrability of (the sub-exponential function) $\Phi(t) = e^{c_0 p x^\rho}$ against $d\mu$ thus gives a hint at quasi-analyticity or not. Still, the function Φ heavily depends on ρ and thus on the space K_Θ . So there is no universal function characterizing quasi-analyticity in terms of the Clark measure as is the case for n -th order derivatives given in (9).

5. UMNK SEQUENCES OF REPRODUCING KERNELS

Lemma 5.1. *If a normalized sequence $\{x_n\}$ is uniformly minimal and contains no Riesz sequences, then $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that*

- (i) $x_{n_k} \xrightarrow{w} x$,
- (ii) $x_{n_k} - x$ is a Riesz sequence,
- (iii) $x \notin \overline{\text{Lin}}\{x_{n_k} - x\}$.

Conversely, any such $\{x_{n_k}\}$ is UMNK.

Proof. We start with the sufficient condition. Since $\{x_n\}$ is uniformly bounded, we can pick a weakly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ (which obviously is UMNK). By Lemma 2.1, $x_{n_k} \xrightarrow{w} x \neq 0$. Since $\{x_{n_k}\}$ is uniformly minimal, no subsequence can converge in norm, so that we can assume $0 < \varepsilon \leq \|x_{n_k} - x\| \leq M$ and hence $\{x_{n_k} - x\}$ can be supposed normalized and $x_{n_k} - x \xrightarrow{w} 0$. Again by Lemma 2.1, and passing possibly to a subsequence we may assume that $\{x_{n_k} - x\}$ is a Riesz sequence.

It remains to check (iii). Since $\{x_{n_k}\}$ is uniformly minimal, there exists a biorthogonal system $\{y_l\}$ such that $\sup_l \|y_l\| < \infty$. Let $z_k = x_{n_k} - x$, which was shown to be a Riesz sequence. Then

$$(z_k, y_l) + (x, y_l) = (z_k + x, y_l) = \delta_{k,l}$$

and, for fixed l and since $\{z_k\}$ is a Riesz sequence, we have $(z_k, y_l) \rightarrow 0$ as $k \rightarrow \infty$. So $(x, y_l) = 0$ and hence $\{y_l\}$ is biorthogonal to $\{z_k\}$. Let $H_0 = \overline{\text{Lin}}\{z_k\}$ and $y_l = y'_l + y''_l$ where $y'_l \in H_0$ and $y''_l \in H_0^\perp$. Then $\{y'_l\}$ is biorthogonal to $\{z_k\}$ in H_0 and

$$(x, y_l) = 0 \iff (x, y'_l) + (x, y''_l) = 0.$$

If $x \in H_0$, then $(x, y''_l) = 0$ and so $(x, y'_l) = 0$. However, $\{y'_l\}$ is the biorthogonal of a Riesz basis in H_0 and thus is a Riesz basis itself in H_0 . Whence $x = 0$ in contradiction to our hypothesis on x . Thus $x \notin H_0$ which shows (iii).

Conversely, suppose $\{x_{n_k}\}$ satisfies (i)–(iii). In particular, by (iii) we have $x \notin H_0$. In the same notation as introduced before, since $\{z_k\}$ is a Riesz basis in H_0 , its biorthogonal $\{y'_l\}$ is also a Riesz basis in H_0 . Clearly we can always find y''_l with bounded norms to get $(x, y''_l) = -(x, y'_l)$, and so the vectors $y_l = y'_l + y''_l$ have uniformly bounded norms and form a biorthogonal system to $\{z_k + x\}$. Hence $\{x_{n_k}\}$ is uniformly minimal. Note that $x \neq 0$ (remember that $x \notin H_0$), and so, (i) and Lemma 2.1 imply that $\{x_{n_k}\}$ cannot contain any Riesz sequence. \square

We state an immediate consequence of the above lemma which we will use in the proof of Theorem 1.8.

Corollary 5.2. *If $\{x_n\}$ is a normalized sequence tending weakly to $x \neq 0$ which has a subsequence not converging in norm to x , then $\{x_n\}$ contains a UMNK sequence.*

Proof. In view of the hypotheses, we can suppose $0 < \varepsilon \leq \|x_{n_k} - x\| \leq M < +\infty$ for some suitable subsequence. Also $x_{n_k} - x \xrightarrow{w} 0$, and by Lemma 2.1, passing to a subsequence, we can suppose that $\{x_{n_k} - x\}$ is a Riesz sequence. This allows us to claim that if $x \in H_0$ then

we can always pass to a subsequence generating a subspace not containing x . It remains to apply Lemma 5.1 to conclude. \square

Proof of Theorem 1.8. Since $\zeta \in \sigma(\theta)$, there exists a sequence $(z_n)_n \subset \mathbb{D}$ converging to ζ such that $\theta(z_n) \rightarrow 0$, $n \rightarrow \infty$. In particular

$$\|k_{z_n}\|^2 = \frac{1 - |\theta(z_n)|^2}{1 - |z_n|^2} \asymp \frac{1}{1 - |z_n|^2} \rightarrow \infty, \quad n \rightarrow \infty.$$

(note that in view of the Ahern–Clark condition, the sequence $(z_n)_n$ has to tend tangentially to ζ). On the other hand, when $\lambda \rightarrow \zeta$ non-tangentially, then, since $\zeta \in AC(\theta)$, $k_\lambda \rightarrow k_\zeta$ in K_θ , in particular $\|k_\lambda\| \rightarrow \|k_\zeta\|$. Thus we may choose a sequence λ_n (on suitable intervals connecting z_n to some fixed Stolz angle at ζ) such that $\lambda_n \rightarrow \zeta$, but

$$\|k_{\lambda_n}\| = 2\|k_\zeta\|.$$

Let us show that $k_{\lambda_n} \xrightarrow{w} k_\zeta$. Indeed, for $g \in K_\theta \cap \overline{C(\mathbb{D})}$ (which, as already mentioned earlier, is a dense subset of K_θ , see [3]),

$$(g, k_{\lambda_n}) = g(\lambda_n) \rightarrow g(\zeta) = (g, k_\zeta).$$

Since the norms $\|k_{\lambda_n}\|$ are bounded, by the Banach–Steinhaus theorem, $k_{\lambda_n} \xrightarrow{w} k_\zeta$. Thus

$$\tilde{k}_{\lambda_n} = \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|} \xrightarrow{w} \frac{k_\zeta}{2\|k_\zeta\|} = \frac{\tilde{k}_\zeta}{2}.$$

and in particular \tilde{k}_{λ_n} has no subsequence converging in norm to \tilde{k}_ζ . By Corollary 5.2, $\{\tilde{k}_{\lambda_n}\}$ is UMR. \square

Remark 5.3. Note that Theorem 1.8 provides a description of those $\lambda_n \rightarrow \zeta$ for which \tilde{k}_{λ_n} is UMR (or contains such system). All we need is that $\sup \|k_{\lambda_n}\| < \infty$ and $\|k_{\lambda_n}\| \not\rightarrow \|k_\zeta\|$.

Before discussing explicit examples of UMR sequences $\{\lambda_n\}$, we briefly discuss the proof of Theorem 1.9.

Proof of Theorem 1.9. By Corollary 2.2 we can suppose that $\sup_n \|k_{z_n}\| < \infty$.

Consider (2). By the above remark, since $\|k_{\lambda_n}\| \not\rightarrow \|k_\zeta\|$, we deduce that \tilde{k}_{λ_n} contains a UMR sequence and in particular a uniformly minimal sequence.

Consider (1). Since ζ is not a point of localization, there exists an infinite sequence $\{z_n\}$ and a non-vanishing function $f \in K_\theta$, such that $f(z_n) = 0$, $n \in \mathbb{N}$. By the backward shift invariance, we can assume that the zeros of f are simple. Then the sequence $\{k_{z_n}\}$ is minimal. Indeed, the sequence $\{\varphi_n\}$ defined by $\varphi_n(z) = f(z)/(z - z_n)$ gives a biorthogonal system. On the other hand $\{k_{z_n}\}$ cannot be uniformly minimal. Indeed, since we are in an Ahern–Clark point, we have $k_{z_n} \rightarrow k_\zeta$, and hence the distance $\|k_{z_n} - k_{z_{n+1}}\|$ goes to zero while $\|k_{z_n}\|$ is uniformly bounded, contradicting thus uniform minimality. \square

Example 5.4. We give an example of a UMR system of normalized reproducing kernels $\{\tilde{k}_{\lambda_n}\}$ having an additional property: the system $\{k_{\lambda_n}\}$ is complete in K_θ . To simplify the estimates we will construct an example in the half-plane setting.

Let $z_n = x_n + iy_n$, $n \geq 1$, be a sequence in \mathbb{C}^+ such that $x_n > 0$, $x_{n+1} > x_n + 1$ and $\sup_n \sum_{k \neq n} |x_n - x_k|^{-1} < \infty$. Furthermore, let $0 < s_n < 1$, put $t_n = x_n + s_n$ for $n \geq 2$, and assume that

$$(12) \quad \frac{y_n}{s_n^2} \asymp \frac{1}{t_n^2}, \quad \sum_{k \neq n} \frac{y_k}{(t_n - x_k)^2} \lesssim \frac{1}{t_n^2}, \quad \sup_n \sum_{k \neq n} \frac{s_k}{|t_k - t_n|} < \infty.$$

Clearly, taking sufficiently small y_n and defining $s_n = x_n \sqrt{y_n}$ we can achieve all the properties.

We will show that under the above assumptions $\{\tilde{k}_{t_n}\}_{n \geq 2}$ is a complete UMR system in K_Θ where Θ is the Blaschke product with zeros z_n .

Define entire functions E and G as zero genus canonical products with zeros $\{\bar{z}_n\}_{n \geq 1}$ and $\{t_n\}_{n \geq 2}$ respectively. Then standard estimates of canonical products (combined with the last inequality in (12)) show that for $z \in \mathbb{C}$ such that $|z - t_n| = \text{dist}(z, \{t_k\})$ we have

$$(13) \quad \left| \frac{G(z)}{E(z)} \right| \asymp \frac{|z - t_n|}{|z - x_n + iy_n|} \cdot \frac{1}{|z| + 1}.$$

Indeed, if $|z - t_n| = \text{dist}(z, \{t_k\})$, then $\sum_{k \neq n} |z - \bar{z}_k|^{-1} \leq C$ for some constant C independent on z and n , whence

$$\sum_{k \neq n} \log \left| \frac{1 - z/t_k}{1 - z/\bar{z}_k} \right| = \sum_{k \neq n} \log \left| 1 + \frac{\bar{z}_k - t_k}{z - \bar{z}_k} \right| + O(1) = O(1).$$

In particular, it follows from (13), that

$$(14) \quad \left| \frac{G'(t_n)}{E(t_n)} \right| \asymp \frac{1}{s_n t_n}.$$

Note that $\Theta(z) := \overline{E(\bar{z})}/E(z)$ is a Blaschke product in \mathbb{C}^+ with zeros z_n . Moreover, the class $E \cdot K_\Theta$ consists of entire functions and coincides with the so-called de Branges space $\mathcal{H}(E)$ (see [14]).

Note that, by (12) and a straightforward estimate,

$$2\pi \|k_{t_n}\|^2 = |\Theta'(t_n)| \leq 2 \sum_k \frac{y_k}{(t_n - x_k)^2} = \frac{2y_n}{s_n^2} + 2 \sum_{k \neq n} \frac{y_k}{(t_n - x_k)^2} \lesssim \frac{1}{t_n^2} \rightarrow 0, \quad n \rightarrow \infty.$$

It follows that $\{\tilde{k}_{t_n}\}$ does not contain any subsequence weakly converging to zero. Indeed, taking $f(z) = (z - \bar{z}_1)^{-1} \in K_\Theta$ we see that $|(f, \tilde{k}_{t_n})| = |f(t_n)|/\|k_{t_n}\| \gtrsim t_n^2 \rightarrow \infty, n \rightarrow \infty$. Thus, by Lemma 2.1, $\{\tilde{k}_{t_n}\}$ does not contain Riesz subsequences. In fact, in the upper half-plane case the condition $\sup_\lambda |\lambda| \cdot \|k_\lambda\|$ is necessary and sufficient for $\{\tilde{k}_\lambda\}$ to contain a Riesz subsequence (compare with Corollary 2.2).

Let us verify that $\{\tilde{k}_{t_n}\}$ is uniformly minimal. It is easy to see that the biorthogonal system to $\{\tilde{k}_{t_n}\}_{n \geq 2}$ is given by

$$g_n(z) = \frac{E(t_n)\|k_{t_n}\|}{G'(t_n)} \cdot \frac{G(z)}{E(z)(z - t_n)}.$$

We need to show that $\sup_n \|g_n\| < \infty$. Let $I_1 = [0, \frac{x_1+x_2}{2}]$ and $I_k = [\frac{x_{k-1}+x_k}{2}, \frac{x_k+x_{k+1}}{2}]$, $k > 1$. Making use of (13) and (14), we see that

$$\begin{aligned} \|g_n\|^2 &\lesssim \int_{I_n} \frac{s_n^2 dx}{((x-x_n)^2 + y_n^2)(x^2+1)} + \sum_{k \neq n} \int_{I_k} \frac{s_n^2 (x-t_k)^2 dx}{((x-x_k)^2 + y_k^2)(x-t_n)^2(x^2+1)} + O(1) \\ &\lesssim \frac{s_n^2}{y_n t_n^2} + s_n^2 \sum_{k \neq n} \int_{I_k} \frac{(x-x_k)^2 + s_k^2}{((x-x_k)^2 + y_k^2)(x-t_n)^2(x^2+1)} dx + O(1) \\ &\lesssim s_n^2 \sum_{k \neq n} \frac{1}{t_k^2 |t_n - t_k|} + s_n^2 \sum_{k \neq n} \frac{s_k^2}{y_k t_k^2 |t_n - t_k|^2} + O(1) = O(1). \end{aligned}$$

In the last inequality we used the last condition in (12) and the fact that $s_n < 1$.

Analogous estimates show that $G/E \notin H^2$. Indeed,

$$\sum_{k=2}^{\infty} \int_{I_k} \left| \frac{G(t)}{E(t)} \right|^2 dt \asymp \sum_{k=2}^{\infty} \int_{I_k} \frac{(x-t_k)^2}{((x-x_k)^2 + y_k^2)(x^2+1)} dx \asymp \sum_{k=2}^{\infty} \frac{s_k^2}{y_k t_k^2}.$$

However, the last series diverges by the first condition in (12).

Finally, we need to show that $\{k_{t_n}\}_{n \geq 2}$ is complete in K_{Θ} . Assume that $h \in K_{\Theta}$ is orthogonal to $\{k_{t_n}\}_{n \geq 2}$, whence $h(t_n) = 0$. Then the entire function $H = Eh$ is divisible by G , i.e., $H = GS$ for some entire function S . We have $S = hE/G$ in \mathbb{C}^+ whence S is in the Smirnov class in \mathbb{C}^+ (see, e.g., [22, Part 2, Chapter 1]) and $|y^{-1}S(iy)| \rightarrow 0$, $y \rightarrow +\infty$, by (13). On the other hand,

$$\overline{S(\bar{z})} = \overline{h(\bar{z})} \cdot \frac{\overline{E(\bar{z})}}{E(z)} \cdot \frac{E(z)}{\overline{G(\bar{z})}},$$

and so S is in the Smirnov class in the lower half-plane and $|y^{-1}S(iy)| \rightarrow 0$, $y \rightarrow -\infty$ (we used the fact that $h \in K_{\Theta}$ and so $\overline{h(\bar{z})} \cdot \overline{E(\bar{z})}/E(z) \in H^2(\mathbb{C}^+)$). By a theorem of M.G. Krein [22, Part II, Chapter 1] S is of zero exponential type and thus the estimates along the imaginary axis imply that S is a constant. If $S \neq 0$, then $G/E \in H^2$, a contradiction.

Remark 5.5. In the above example the constructed UMNR system is also complete in K_{θ} . Do such examples exist in the general case? Namely, assume that $\sigma(\theta)$ consist of one point (or of finite number of points). Does there exist a complete UMNR system of reproducing kernels?

Remark 5.6. In Example 1 the points were chosen on the real axis. This is not always possible. E.g., if $z_n = n + in^{-3/2}$, $n \in \mathbb{N}$, then ∞ is an Ahern–Clark point for the corresponding Blaschke product Θ , but $t^2|\Theta'(t)| \rightarrow \infty$, $t \rightarrow \infty$. Hence, any minimal system of normalized reproducing kernels $\{\tilde{k}_{t_n}\}$ contains a Riesz subsequence.

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